

# A CRITERION FOR GLOBAL DIMENSION TWO FOR STRONGLY SIMPLY CONNECTED SCHURIAN ALGEBRAS

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**ABSTRACT.** The aim of this paper is to provide a criterion to determine, by quivers with relations, when an algebra has global dimension at most two. In order to do that, we introduce a new class of algebras of global dimension three, and we call them critical algebras. Furthermore we give a characterization of critical algebras by quivers with relations. Our main theorem states that if a strongly simply connected schurian algebra does not contain a critical algebra as a full subcategory, then it has global dimension at most two.

## 1. INTRODUCTION

The algebras of global dimension two play an important roll in the representation theory of finite dimensional algebras, for example, two remarkable classes of algebras of global dimension two are tilted and quasitilted algebras. On other hand, in [1], Amiot, introduced a new cluster category associated with an algebra of global dimension two. Hence the interest for algebras of global dimension two has revived recently in connection with cluster categories.

An interesting problem is to find a criterion, by quivers with relations, to determine when an algebra has global dimension two. Some criterions of this type have been given in the literature, a criterion was given by Green, Happel and Zacharia, in [9], for monomial algebras, and by Igusa and Zacharia, in [10], in the case of incidence algebras.

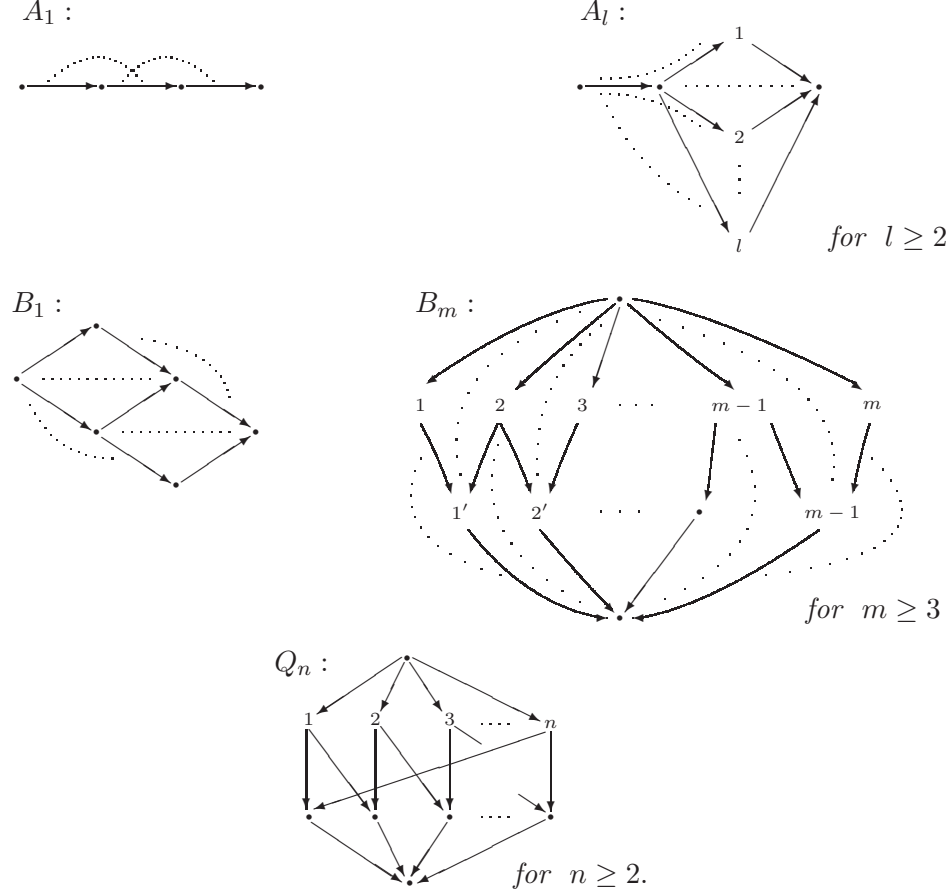
In this work we provide a criterion for a well known class of algebras which are quotients of incidence algebras. We recall that a subcategory  $B$  of  $A$  is called *full* if  $\text{Hom}_B(Si, Sj) = \text{Hom}_A(Si, Sj)$ , for all  $i, j \in (Q_B)_0$ . In order to find the desired criterion, we introduce a new family of algebras and we call them critical algebras. These algebras have global dimension three and have the property that every proper full subcategory has global dimension two. We give here a characterization of critical algebras in terms of quivers with relations.

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**Proposition.** Let  $\Gamma$  be a critical algebra. Then either  $\Gamma$  or  $\Gamma^{op}$  is one of the following algebras.



Before stating our main theorem, we need to recall some definitions. A full subcategory  $B$  of  $A$  is called *convex* if any path in  $A$  with source and target in  $B$  lies entirely in  $B$ . An algebra  $A$  is called *triangular* if  $Q_A$  has no oriented cycles, and it is called *schurian* if, for all  $x, y \in A_0$ , we have  $\dim_k A(x, y) \leq 1$ . A triangular algebra  $A$  is called *simply connected* if, for any presentation  $(Q_A, I)$  of  $A$ , the group  $\pi_1(Q_A, I)$  is trivial, see [5]. It is called *strongly simply connected* if every full convex subcategory of  $A$  is simply connected, [11].

Now, we are in a position to state our main theorem.

**Theorem.** Let  $A$  be a strongly simply connected schurian algebra. Then if  $A$  does not contain a critical algebra as a full subcategory, it follows that  $\text{gl.dim. } A \leq 2$ .

The converse of the theorem does not hold.

The paper is organized in the following way. In Section 2 we introduce some preliminary concepts and notations. In Section 3 we construct minimal projective resolutions for the simple modules over a strongly simply connected schurian algebra and we compute its projective dimension. In Section 4 we introduce critical algebras and we characterize them by quivers with relations. Finally, we establish the main theorem of this paper.

## 2. PRELIMINARIES

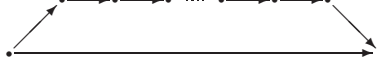
**2.1. Notation.** In this paper, by algebra, we always mean a basic and connected finite dimensional algebra over an algebraically closed field  $k$ . Given a quiver  $Q$ , we denote by  $Q_0$  its set of vertices and by  $Q_1$  its set of arrows. A *relation* in  $Q$  from a vertex  $x$  to a vertex  $y$  is a linear combination  $\rho = \sum_{i=1}^m \lambda_i w_i$  where, for each  $i$ ,  $\lambda_i \in k$  is non-zero and  $w_i$  is a path of length at least two from  $x$  to  $y$ . A relation in  $Q$  is called a *monomial* if it equals a path, and a *commutativity relation* if it equals the difference of two paths. A relation  $\rho$  is called *minimal* if whenever  $\rho = \sum_i \beta_i \rho_i \gamma_i$  where  $\rho_i$  is a relation for every  $i$ , then  $\beta_i$  and  $\gamma_i$  are scalars for some index  $i$  (see [7]).

We denote by  $kQ$  the path algebra of  $Q$  and by  $kQ(x, y)$  the  $k$ -vector space generated by all paths in  $Q$  from  $x$  to  $y$ . For an algebra  $A$ , we denote by  $Q_A$  its quiver. For every algebra  $A$ , there exists an ideal  $I$  in  $kQ_A$ , generated by a set of relations, such that  $A \simeq kQ_A/I$ . The pair  $(Q_A, I)$  is called a *presentation* of  $A$ . An algebra  $A = kQ/I$  can equivalently be considered as a  $k$ -category of which the object class  $A_0$  is  $Q_0$ , and the set of morphisms  $A(x, y)$  from  $x$  to  $y$  is the quotient of  $kQ(x, y)$  by the subspace  $I(x, y) = I \cap kQ(x, y)$ .

In this work, we always deal with schurian triangular algebras. For a vertex  $x$  in the quiver  $Q_A$ , we denote by  $e_x$  the corresponding primitive idempotent,  $S_x$  the corresponding simple  $A$ -module, and by  $P_x$  and  $I_x$  the corresponding indecomposable projective and injective  $A$ -module, respectively.

Let  $Q$  be a connected quiver without oriented cycles. A *contour*  $(p, q)$  in  $Q$  from  $x$  to  $y$  is a pair of parallel paths of positive length from  $x$  to  $y$ . A contour  $(p, q)$  is called *interlaced* if  $p$  and  $q$  have a common vertex besides  $x$  and  $y$ . It is called *irreducible* if there exists no sequence of paths  $p = p_0, p_1, \dots, p_m = q$  from  $x$  to  $y$  such that, for each  $i$ , the contour  $(p_i, p_{i+1})$  is interlaced.

**2.2. Incidence algebras and their quotients.** Let  $(\Sigma, \leq)$  be a finite poset (partially ordered set) with  $n$  elements. The incidence algebra  $k\Sigma$  is the subalgebra of the algebra  $M_n(k)$  of all  $n \times n$  matrices over  $k$  consisting of the matrices  $[a_{ij}]$  satisfying  $a_{ij} = 0$  if  $j \not\leq i$ . The quiver  $Q_\Sigma$  of  $k\Sigma$  is the (oriented) Hasse diagram of  $\Sigma$ , and  $k\Sigma \simeq Q_\Sigma/I_\Sigma$ , where  $I_\Sigma$  is generated by all differences  $p - q$ , with  $(p, q)$  a contour in  $Q_\Sigma$ . The quiver  $Q\Sigma$  has no bypass, that is, no subquiver of the form



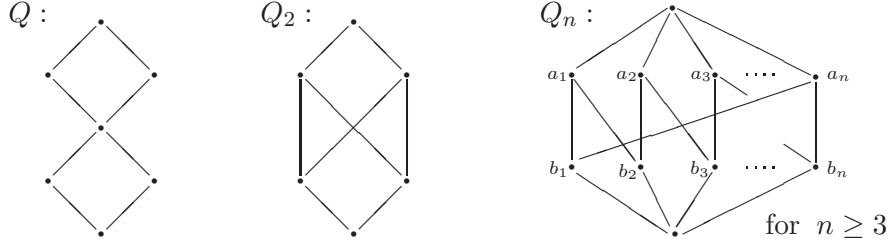
and, conversely, for any quiver  $Q$  having no bypass, there exists a poset  $\Sigma$  such that  $Q = Q_\Sigma$ .

If  $A$  is an incidence algebra and  $x \in (Q_A)_0 = A_0$ , then it is easy to see that the simple modules  $S_x$ , indecomposable projective  $P_x$  and indecomposable injective  $I_x$ , are described, as representation, as follows:

- $S_x$  is given by  $S_x(x) = k$  and  $S_x(y) = 0$  for  $y \neq x$ , plus  $S_x(\alpha) = 0$ , for any arrow  $\alpha$ .
- $P_x$  is given by  $P_x(y) = k$  if  $x \geq y$  and  $P_x(y) = 0$  in another case, plus  $P_x(\alpha) = 1$  if  $x \geq s(\alpha)$  and  $P_x(\alpha) = 0$  otherwise.
- $I_x$  is constructed dually to  $P_x$ .

Note that any incidence algebra  $A = A(\Sigma)$ , the full subcategory (or full convex) of  $A$  coincide with the incidence algebras of the full subposet (or full convex) of  $\Sigma$ .

In [10] is shown that, if  $A = k\Sigma$  is an incidence algebra, then  $\text{gl.dim. } A \leq 2$  iff  $\Sigma$  does not contain a full subposet isomorphic to  $Q_n$  ( $n \geq 3$ ) and all full subposet of  $\Sigma$  isomorphic to  $Q_2$  is contained in a full subposet  $Q$  of  $\Sigma$ , where



We are going to consider quotients of incidence algebras. For such a quotient  $A \simeq kQA/I$ , there exists a poset  $\Sigma$  with  $Q_\Sigma = Q_A$  and, furthermore,  $I = I_\Sigma + J$ , where  $J$  is an ideal of  $kQ_\Sigma$  generated by monomials. It is well known that, if  $A$  is schurian strongly simply connected, then it is a quotient of an incidence algebra, see [8], [3].

Conversely, in [2], the authors proved that, if we have a poset  $\Sigma$  such that  $k\Sigma$  is strongly simply connected, and we consider  $A \cong k\Sigma/J$ , where  $J$  is an ideal of  $k\Sigma$  generated paths are not completely contained in irreducible contours, then  $A$  is strongly simply connected algebra.

The above results allow us to describe the indecomposable projective modules of a strongly simply connected schurian algebra. Indeed, let  $A = kQ_\Sigma/I_\Sigma + J$  be a strongly simply connected schurian algebra and let  $x \in (Q_A)_0$ . Let  $\rho_1, \dots, \rho_r$  be minimal relations in  $J$  such that  $x \geq s(\rho_i)$ , for all  $1 \leq i \leq r$ . Then:

$$P_x^A(z) = \begin{cases} k & \text{if } x \geq z, z \not\geq t(\rho_i) \forall 1 \leq i \leq r, \\ 0 & \text{otherwise,} \end{cases}$$

with the induced morphisms.

### 3. ON THE PROJECTIVE DIMENSION OF SIMPLE MODULES

In this Section we consider strongly simply connected schurian algebras. Our main objective is to describe the first terms of the minimal projective resolution of a simple module. As a consequence we are able to study the projective dimension of the simple modules.

The following remark is important for our purposes.

**Remark 3.1.** Let  $A = kQ/I$  be a strongly simply connected schurian algebra. Let

$$(3.1) \quad \cdots \longrightarrow Q_3 \xrightarrow{f_3} Q_2 \xrightarrow{f_2} Q_1 \xrightarrow{f_1} Q(S_i) = P_i \xrightarrow{f_0} S_i \longrightarrow 0$$

be the minimal projective resolution of  $A$ -module simple  $S_i$ .

- (1) The projective  $Q_1$  is the direct sum of indecomposable projective  $P_a$ , where there is an arrow from the vertex  $i$  to the vertex  $a$ .
- (2) If  $P_j$  is a direct summand of the projective  $Q_{r+1}$  ( $r \geq 2$ ), then  $S_j \in \text{Top } Q_{r+1} = \text{Top Ker } f_r$  and  $S_j$  is a composition factor of  $Q_r$ . Moreover, if  $S_j$  is not a composition factor of  $\text{Ker } f_{r-1}$ , then for all  $h$  such that  $S_h \in \text{Top } Q_{r-1}$ , it follows that all paths from  $h$  to  $j$  are zero paths in  $A$ .

$$\begin{array}{ccccccc} \cdots & \longrightarrow & Q_{r+1} & \xrightarrow{f_{r+1}} & Q_r & \xrightarrow{f_r} & Q_{r-1} \xrightarrow{f_{r-1}} \cdots \\ & & \searrow & & \swarrow & & \swarrow \\ & & & \text{Ker } f_r & & \text{Ker } f_{r-1} & \end{array}$$

- (3) Repeating the process, we obtain that, for any indecomposable projective  $A$ -module  $P_d$  which is a direct summand of any term of (3.1), we have that there exists a path from  $i$  to  $d$  in the quiver  $Q$ .

From Remark 3.1, we get a description of the first and second terms of the minimal projective resolution (3.1). We continue studying the behavior of some of the other terms of this resolution.

From now on, we denote by  $\mu_M(S)$  the multiplicity of the simple module  $S$  as a composition factor of the  $A$ -module  $M$ .

The following proposition describes the term  $Q_2$ .

**Proposition 3.2.** *Let  $A = kQ/I$  be a strongly simply connected schurian algebra. Let  $i$  and  $b$  be vertices of  $Q$ , and let*

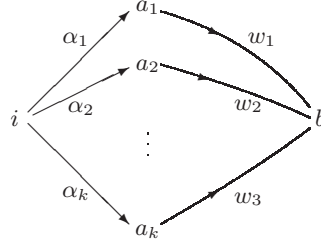
$$\cdots \longrightarrow Q_3 \xrightarrow{f_3} Q_2 \xrightarrow{f_2} Q_1 \xrightarrow{f_1} Q_0 \xrightarrow{f_0} S_i \longrightarrow 0$$

be the minimal projective resolution of simple  $A$ -module  $S_i$ . Then the following statements are equivalent:

- (1)  $P_b$  is a direct summand of  $Q_2$ ,
- (2) there is a minimal relation from  $i$  to  $b$ .

*Proof.* If  $P_b$  is a direct summand of  $Q_2$ , then there exist a vertex  $a_1 \neq b$  such that  $P_{a_1}$  is a direct summand of  $Q_1$  and a nonzero path  $w_1 : a_1 \rightsquigarrow b$  in the algebra  $A$ .

Let  $a_1, a_2, \dots, a_k$  all vertices that satisfy:  $P_{a_j}$  is a direct summand of  $Q_1$ ,  $w_j : a_j \rightsquigarrow b$  is a nonzero path in  $A$ , for  $j = 1, \dots, k$ . We denote  $\alpha_j$  the corresponding arrow from the vertex  $i$  to vertex  $a_j$ . Then, in the quiver  $Q$ , we have the following situation:



If  $\alpha_j w_j = 0$ , for all  $j = 1, \dots, k$ , then there are monomial relations  $\rho_1, \rho_2, \dots, \rho_k$  in  $I$ , such that  $s(\rho_j) = i$  and  $t(\rho_j) = q_j$ , for all  $j = 1, \dots, k$ , where  $a_j > q_j \geq b$  and  $q_j$  is a vertex of the path  $w_j$ . We can assume that  $\rho_1, \rho_2, \dots, \rho_k$  are minimal relations in  $I$ .

If some  $q_j = b$ , we obtain the result. Suppose that  $q_j \neq b$ , for all  $j = 1, \dots, k$ . Then, for each  $j = 1, \dots, k$ , it follows that  $S_{q_j}$  is not a composition factor of  $\text{rad } P_i$  and  $S_{q_j} \in \text{Top Ker } f_1 = \text{Top } Q_2$ , which is a contradiction. Therefore, there exists a minimal relation that starts at  $i$  and ends at  $b$ .

Now, if for some  $1 \leq j \leq k$ ,  $\alpha_j w_j \neq 0$ , since  $A$  is a quotient of an incidence algebra, it follows that all parallel paths to  $\alpha_j w_j$  are nonzero. In particular, we get that  $\alpha_{j+1} w_{j+1} \neq 0$  and also  $\alpha_j w_j - \alpha_{j+1} w_{j+1} \in I$ . Then there is a commutativity relation from  $i$  to  $b$ . A similar argument shows that some of these  $k$  commutativity relations must be minimal.

Conversely, let  $\rho : i \rightsquigarrow b$  be a minimal relation. Then we have again a similar situation to the one described in the previous figure, where the paths  $w_1, w_2, \dots, w_k$  could be zero paths in the algebra  $A$ .

If  $\rho$  is a monomial relation, then we can assume that  $\rho = \alpha_1 w_1 = 0$  in  $A$  and  $w_1 \neq 0$  in  $A$ . Since  $A$  is a quotient of an incidence algebra, it follows that  $\alpha_j w_j = 0$  in  $A$ , for  $1 \leq j \leq k$ , and we get that  $S_b \notin \text{rad } P_i$ . Moreover, since  $\rho$

is a minimal relation, it must be  $S_b \in \text{Top Ker } f_1 = \text{Top } Q_2$ . Consequently,  $P_b$  is a direct summand of  $Q_2$ .

Now suppose that  $\rho$  is a minimal commutativity relation. Without loss of generality, we can assume that  $\rho = \alpha_1 w_1 - \alpha_2 w_2$ , with  $\alpha_1 w_1 \neq 0$ ,  $\alpha_2 w_2 \neq 0$ . Then  $S_b$  is a composition factor of  $P_i$ . Then, since  $A$  is schurian,  $\mu_{\text{rad } P_i}(S_b)$  is either 0 or 1. Since  $\mu_{P_i}(S_b) = k$ , then  $\mu_{\text{Ker } f_1}(S_b) \geq 1$ . Hence the minimality of  $\rho$  implies that  $S_b \in \text{Top Ker } f_1$ ; i.e.,  $P_b$  is a direct summand of  $Q_2$ .  $\square$

Next, we recall the notion of convex hull of two vertices and some useful results.

Let  $A = kQ/I$  be an algebra and let  $i, j$  be vertices of  $Q$ . The *convex hull* between  $i$  and  $j$ ,  $\text{Conv}(i, j) = kQ'/I'$  is the subalgebra of  $A$  given by the quiver  $Q'$

- $(Q')_0 = \{k \in Q_0 / \text{there are walks } i \rightsquigarrow k \rightsquigarrow j\}$
- $(Q')_1 = \{\alpha \in Q_1 / s(\alpha) \text{ and } t(\alpha) \in (Q')_0\}$

and  $I'$  is generated by induced relations.

Note that  $C = \text{Conv}(i, j)$ , as  $k$ -category is a full and convex subcategory of  $A$ . Under these conditions, it follows from [4] that  $\text{Ext}_A^i(X, Y) \cong \text{Ext}_C^i(X, Y)$  for all  $i \geq 0$  and  $X, Y \in \text{mod } C$ .

It follows that, if the algebra  $A$  is strongly simply connected and schurian, then so is  $C = \text{Conv}(i, j)$ .

**Lemma 3.3.** *Let  $A = kQ/I$  be a strongly simply connected schurian algebra, let  $i$  and  $j$  be vertices of  $Q$ , and let*

$$\cdots \longrightarrow Q_3 \xrightarrow{f_3} Q_2 \xrightarrow{f_2} Q_1 \xrightarrow{f_1} Q_0 \xrightarrow{f_0} S_i \longrightarrow 0$$

*be the minimal projective resolution of the simple  $A$ -module  $S_i$ . If  $P_j$  is a direct summand of  $Q_3$ , then  $S_j$  is not a composition factor of the term  $Q_0$ .*

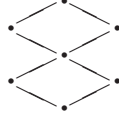
*Proof.* Suppose  $S_j$  is a composition factor of the term  $Q_0$ , where  $Q_0$  is the indecomposable projective module associated with the vertex  $i$ . Since the algebra  $A$  is schurian, we have that  $\mu_{Q_0}(S_j) = 1$ . Then in  $C = \text{Conv}(i, j)$  there is no monomial relations. Therefore,  $C$  is an incidence algebra. Moreover, since  $S_j \in \text{Top } Q_3$ , there exists a morphism  $\pi : Q_3 \longrightarrow S_j$ . Considering the push out of  $\pi$  and  $f_3$  we have the following commutative diagram:

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & Q_3 & \xrightarrow{f_3} & Q_2 & \xrightarrow{f_2} & Q_1 & \xrightarrow{f_1} & Q_0 & \xrightarrow{f_0} & S_i & \longrightarrow & 0 \\ & & \downarrow \pi & & \downarrow & & \downarrow Id & & \downarrow Id & & \downarrow Id & & \\ 0 & \longrightarrow & S_j & \longrightarrow & Q' & \longrightarrow & Q_1 & \longrightarrow & Q_0 & \longrightarrow & S_i & \longrightarrow & 0 \end{array}$$

with exact rows.

Since  $S_j \in \text{Top Ker } f_2$ , it follows that  $\pi$  does not factor through  $Q_2$ ; and consequently,  $\text{Ext}_A^3(S_i, S_j) \neq 0$ . Since  $\text{Ext}_C^3(S_i, S_j) \cong \text{Ext}_A^3(S_i, S_j)$ , we have that  $\text{dp}_C S_i = 3$ .

Then,  $C$  is an incidence algebra with global dimension at least three. By [10], either  $C = k\Sigma$  must contain some graph  $Q_n$ , for  $n \geq 3$ , or there is a subposet  $\Sigma'$  of  $\Sigma$  isomorphic to  $Q_2$  that is not contained in any subposet of the form



In both cases  $C$  is not strongly simply connected. Therefore, there is a monomial relation in  $C$ , and we get that  $S_j$  is not a composition factor of  $Q_0$ .  $\square$

From now on, we consider the convex hull  $C = \text{Conv}(i, j) \cong kQ_C/I_C$  between two vertices  $i$  and  $j$  such that:

- $\text{pd}_C S_k < \text{pd}_C S_i$ , for all  $k \in (Q_C)_0$ .
- $\text{Ext}_A^3(S_i, S_j) \neq 0$ , and if  $\text{Ext}_A^3(S_i, S_k) \neq 0$  for  $k \in (Q_C)_0$ ,  $k \neq j$ , then  $k \not\prec j$ .

**Lemma 3.4.** *Let  $A = kQ/I$  be a strongly simply connected schurian algebra. Let  $S_i$  be a simple  $A$ -module with  $\text{pd}_A S_i = 3$  and let*

$$0 \longrightarrow Q_3 \xrightarrow{f_3} Q_2 \xrightarrow{f_2} Q_1 \xrightarrow{f_1} Q_i \xrightarrow{f_0} S_i \longrightarrow 0$$

*be the minimal projective resolution of  $S_i$  in  $A$ . Let  $P_j$  be a direct summand  $Q_3$  and  $C = \text{Conv}(i, j)$ . Then,  $\text{dp}_C S_i = 3$  in  $C$ .*

*Proof.* From the previous lemma, it follows that  $\text{Ext}_A^3(S_i, S_j) \neq 0$  and  $\text{Ext}_A^4(S_i, S_j) = 0$ . Since  $C$  is a full and convex subcategory of  $A$ , we have that  $\text{Ext}_C^3(S_i, S_j) \neq 0$  and  $\text{Ext}_C^4(S_i, S_j) = 0$ . Therefore,  $\text{dp}_C S_i = 3$ .  $\square$

**Proposition 3.5.** *Let  $A = kQ/I$  be a strongly simply connected schurian algebra. Let  $i$  and  $j$  be vertices of  $Q$ , and let*

$$\dots \longrightarrow Q_3 \xrightarrow{f_3} Q_2 \xrightarrow{f_2} Q_1 \xrightarrow{f_1} Q_0 \xrightarrow{f_0} S_i \longrightarrow 0$$

*be the minimal projective resolution of  $S_i$  in  $A$ . Consider the following set of vertices:*

- $\mathcal{R} = \{b \in Q_0 : P_b \text{ is a direct summand of } Q_2 \text{ and there is a nonzero path } b \rightsquigarrow j\}$ ,



- $\mathcal{S} = \{a \in Q_0 : P_a \text{ is a direct summand of } Q_1 \text{ and there is } b \in \mathcal{R} \text{ such that } a \rightsquigarrow b \text{ is a nonzero path}\},$
- $r = \text{Card } \mathcal{R} \text{ and } s = \text{Card } \mathcal{S}.$

If  $s \geq r$ , then the following statements are equivalent:

- (1)  $P_j$  is a direct summand of  $Q_3$ ,
- (2) there are at least  $s - r + 1$  monomial relations  $a \rightsquigarrow j$  with  $a \in \mathcal{S}$  and at least one (monomial or commutative) relation from  $a \in \mathcal{S}$  to the vertex  $j$  is a minimal relation.

*Proof.* Suppose that  $P_j$  is a direct summand of  $P_3$ ,  $\mathcal{S} = \{a_1, \dots, a_s\}$  and  $\mathcal{R} = \{b_1, \dots, b_r\}$ . If  $\mu_{P_{a_k}}(S_j) = 1$ , for all  $1 \leq k \leq s$ , then

$$\mu_{\text{Ker}f_1}(S_j) = \mu_{Q_1}(S_j) = \sum_{k=1}^s \mu_{P_{a_k}}(S_j) = s \geq r = \sum_{h=1}^r \mu_{P_{b_h}}(S_j) = \mu_{Q_2}(S_j),$$

since  $\mu_{P_i}(S_j) = 0$ . Since  $\mu_{Q_2}(S_j) = \mu_{\text{Ker}f_1}(S_j) + \mu_{\text{Ker}f_2}(S_j)$ , it follows that  $\mu_{\text{Ker}f_1}(S_j) = s = r = \mu_{Q_2}(S_j)$ . This implies that  $\mu_{\text{Ker}f_2}(S_j) = 0$ , a contradiction. Therefore, there is  $a \in \mathcal{S}$  such that  $\mu_{P_a}(S_j) = 0$ .

Let  $\mu_{Q_1}(S_j) = \mu_{\text{Ker}f_1}(S_j) = q$ , i.e., there are  $s - q$  monomial relations  $a \rightsquigarrow j$  with  $a \in \mathcal{S}$ . Let  $\mu_{Q_3}(S_j) = \mu_{\text{Ker}f_2}(S_j) = \alpha > 0$ . Then

$$r = \mu_{Q_2}(S_j) = \mu_{\text{Ker}f_1}(S_j) + \mu_{\text{Ker}f_2}(S_j) = q + \alpha,$$

implies that  $q < r$ . Then,  $s - q > s - r \geq 0$ . As a consequence, we get that there are at least  $s - r + 1$  monomial relations  $a \rightsquigarrow j$  with  $a \in \mathcal{S}$ . Also, if none of the relations from  $a \in \mathcal{S}$  to  $j$  is minimal, then  $S_j$  can not be in  $\text{Top Ker } f_2$ .

Conversely, if there are  $q \geq s - r + 1$  monomial relations  $a \rightsquigarrow c$  with  $a \in \mathcal{S}$ , then

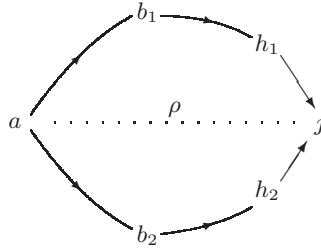
$$\mu_{\text{Ker}f_1}(S_j) = \mu_{Q_1}(S_j) = s - q \geq r - 1.$$

Since  $\mu_{Q_2}(S_j) = r$ , it follows that

$$r = \mu_{Q_2}(S_j) = \mu_{\text{Ker}f_1}(S_j) + \mu_{\text{Ker}f_2}(S_j) \leq r - 1 + \mu_{\text{Ker}f_2}(S_j).$$

We get that,  $1 \leq \mu_{\text{Ker}f_2}(S_j) \leq \mu_{Q_3}(S_j)$ .

Since at least one of the (or monomial or commutative) relation starting at the vertex  $a \in \mathcal{S}$  and ending in the vertex  $j$  is a minimal relation, we have that  $S_j \in \text{Top Ker } f_2$ . Indeed, if the minimal relation  $\rho : a \rightsquigarrow b \rightsquigarrow h \longrightarrow c$  is monomial, then  $\mu_{P_a}(S_h) = 1$ ,  $\mu_{P_a}(S_j) = 0$ ,  $\mu_{P_b}(S_h) = 1$ ,  $\mu_{P_b}(S_j) = 1$ . Now, if the minimal relation is a commutative relation



we have that  $\mu_{P_a}(S_{h_1}) = 1 = \mu_{P_a}(S_{h_2})$ ,  $\mu_{P_a}(S_j) = 1$ ,  $\mu_{P_{b_1}}(S_{h_1}) = 1 = \mu_{P_{b_2}}(S_{h_2}) = 1$ ,  $\mu_{P_{b_1}}(S_j) = 1 = \mu_{P_{b_2}}(S_j)$ . In both cases, it is clear that  $S_j$  must belong to  $\text{Top Ker } f_2$ . Then  $P_j$  is a direct summand of  $Q_3$ .  $\square$

Note that, without loss of generality, we can take  $s \geq r$ . Otherwise, the opposite algebra satisfies the desired property.

Our purpose now is to study the projective dimensions of the simple modules. To do this, we need the following technical lemmas.

**Lemma 3.6.** *Let  $A = kQ/I$  be a strongly simply connected schurian algebra. Let*

$$0 \longrightarrow Q_n \xrightarrow{f_n} \cdots \xrightarrow{f_2} Q_1 \xrightarrow{f_1} Q_0 \xrightarrow{f_0} S_i \longrightarrow 0$$

*be the minimal projective resolution of simple  $A$ -module  $S_i$ . Then  $S_i$  is not a composition factor of the  $A$ -modules  $\text{rad } Q_0, Q_1, \dots, Q_n$ .*

*Proof.* If  $S_i \in \text{rad } Q_0 = \text{Ker } f_0$ , then there exists a path of length at least one from  $i$  to  $i$  in  $Q$ , which is a contradiction since  $A$  is triangular.

Suppose now that  $S_i$  appears as a composition factor in some term  $Q_j$ , with  $j \geq 1$ . Let  $0 \neq k = \min\{l \in \{1, \dots, n\} \text{ such that } \mu_{P_l}(S_i) \neq 0\}$ . Then  $\mu_{\text{Ker } f_{k-1}}(S_i) = 0$  and there is  $S_k$  in  $\text{Top } Q_k = \text{Top Ker } f_{k-1}$ , such that:

- (k) there is a nonzero path from  $k$  to  $i$
- (k-1) there is  $S_{k-1} \in \text{Top } Q_{k-1} = \text{Top Ker } f_{k-2}$  such that there is a nonzero path from  $(k-1)$  to  $k$

Consider  $0 \leq h \leq k-1$ :

- (k-h) there is  $S_{k-h} \in \text{Top } Q_{k-h} = \text{Top Ker } f_{k-h-1}$  such that there is a non-zero path  $w_h$  from  $k-h$  to  $k-h+1$

We obtain  $S_1 \in \text{Top Ker } f_0 = \text{Top rad } Q_0$ , and therefore there is an arrow from  $i$  to 1. Consequently,

$$i \longrightarrow 1 \rightsquigarrow \cdots \rightsquigarrow (k-h) \rightsquigarrow (k-1) \rightsquigarrow k \rightsquigarrow i$$

is a cycle in  $Q$ , which is a contradiction since  $A$  is triangular.  $\square$

**Lemma 3.7.** *Let  $A = kQ/I$  a strongly simply connected schurian algebra with  $\text{gl.dim. } A = n$ . Consider  $J = \{h \in Q_0 : \text{pd}_A S_h = n\}$ . Then there exists  $j \in J$  such that  $S_h$  do not appear in the minimal projective resolution of  $S_j$ , with  $h \in J \setminus \{j\}$ .*

*Proof.* Suppose that, for all  $j \in J$ , there is  $h \in J$  such that  $S_h$  appears as a composition factor in the minimal projective resolution of  $S_j$ . Let  $Q_k$  be the first term of the minimal projective resolution of  $S_j$  in which  $S_h$  appears as

composition factor, then  $\mu_{\text{Ker}f_{k-1}}(S_h) = 0$ . An analogous argument to the one used in Lemma 3.6, shows that there is a path  $j \rightsquigarrow h$  and also  $h \neq j$ .

Next, we will renumber the elements of  $J$ . Let  $j \in J$  and  $j_1 \in J \setminus \{j\}$  such that  $S_{j_1}$  appears in the minimal projective resolution of  $S_j$ . Now we consider  $j_2 \in J \setminus \{j_1\}$  such that  $S_{j_2}$  appears in the minimal projective resolution of  $S_{j_1}$ . If  $j_2 = j$  we have a path in  $Q$ . It follows that  $j_2 \neq j$  and there is  $j_3 \in J \setminus \{j, j_1, j_2\}$  such that  $S_{j_3}$  appears in the minimal projective resolution of  $S_{j_2}$ . Therefore,

$$J = \{j, j_1, \dots, j_{m-1}\}$$

and there exists a sequence of paths in  $Q$ :

$$j \rightsquigarrow j_1 \rightsquigarrow j_2 \rightsquigarrow \dots \rightsquigarrow j_{m-1}$$

Since  $S_h$  appears in the minimal projective resolution of  $S_{j_{m-1}}$ , for some  $h \in J$ , we get that either  $h = j$ , or  $h = j_l$ , with  $1 \leq l \leq m-1$ , are a contradiction with  $A$  triangular.

Consequently, there is a  $j \in J$  such that any  $S_h, h \in J \setminus \{j\}$ , does not appear as composition factor in the minimal projective resolution of  $S_j$ .  $\square$

**Proposition 3.8.** *Let  $A$  be an algebra with  $\text{gl.dim. } A = n$  and let  $M$  be an  $A$ -module. If  $\text{pd } M = n$ , then there is a composition factor  $S$  of  $M$ , such that  $\text{pd } S = n$ .*

*Proof.* The proof follows by induction on the Loewy length  $l_w(M)$  of  $M$ .

If  $l_w(M) = 1$  then  $M$  is semisimple, and it follows that one of its direct summands has projective dimension  $n$ . Suppose now that the result holds for  $l_w(M) < m$ . Consider the exact sequence

$$0 \rightarrow \text{rad } M \rightarrow M \rightarrow \text{Top } M \rightarrow 0$$

then we have that  $l_w(\text{rad } M) = m - 1$  and  $n = \text{pd } M \leq \sup\{\text{pd } \text{rad } M, \text{pd } \text{Top } M\}$ .

Since  $\text{gl.dim. } A = n$ , then  $\sup\{\text{pd } \text{rad } M, \text{pd } \text{Top } M\} \leq n$ . Therefore  $n = \text{pd } M = \sup\{\text{pd } \text{rad } M, \text{pd } \text{Top } M\}$ .

If  $\text{pd } \text{Top } M = n$ , then since the module  $\text{Top } M$  is semisimple, the result follows. Otherwise,  $\text{pd } \text{rad } M = n$ , and by inductive hypothesis there is  $S$  composition factor  $\text{rad } M$  such that  $\text{pd } S = n$ . Since  $S$  is also a composition factor of  $M$ , the result follows.  $\square$

Now, we are able to state the following result.

**Theorem 3.9.** *Let  $A = kQ/I$  be a strongly simply connected schurian algebra with  $\text{gl.dim. } A = n$ . Then, for all  $0 \leq m \leq n$ , there is a simple  $A$ -module  $S$  such that  $\text{pd}_A S = m$ .*

*Proof.* We consider again the set  $J = \{h \in Q_0 : pd_A S_h = n\}$ . As  $n = \text{gl.dim. } A = \sup\{pd_A S_i : i \in Q_0\}$ , it follows that  $J \neq \emptyset$ .

Since  $A$  is triangular, it follows that the algebra  $B = A/\langle\{e_k : k \in J\}\rangle$  is also triangular, and we have the functorial immersion  $\text{mod } B \hookrightarrow \text{mod } A$ .

By Lemma 3.7, there is  $j \in J$  such that if

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Q_n & \longrightarrow & \cdots & \longrightarrow & Q_1 & \longrightarrow & Q_0 & \longrightarrow & S_j & \longrightarrow & 0 \\ & & & & & & \searrow & & \nearrow & & & & \\ & & & & & & & \text{rad } Q_0 & & & & & \end{array}$$

is the minimal projective resolution of  $S_j$ , then  $\mu_{\bigoplus_{t=0}^n Q_t}(S_k) = 0$ , for all  $k \in J \setminus \{j\}$ . Hence  $0 \longrightarrow Q_n \longrightarrow \cdots \longrightarrow Q_1 \longrightarrow \text{rad } Q_0 \longrightarrow 0$  is the projective resolution of the module  $\text{rad } Q_0$  in  $A$  and also in  $B$ , because any  $S_k$ ,  $k \in J$ , does not appear as a composition factor of the projective modules  $Q_i$ ,  $1 \leq i \leq n$ . Therefore,  $pd_A \text{rad } Q_0 = pd_B \text{rad } Q_0 = n - 1$ .

Let  $h \notin J$ , the simple  $B$ -module  $S_h$  is also a simple  $A$ -module with  $s = pd_A S_h \leq n - 1$ .

Consider the projective resolutions of  $S_h$  in  $\text{mod } A$  and  $\text{mod } B$ , respectively.

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & Q_s^A & \longrightarrow & Q_{s-1}^A & \longrightarrow & \cdots & \longrightarrow & Q_1^A & \longrightarrow & Q_0^A & \xrightarrow{f_0^A} & S_h & \longrightarrow & 0 \\ & & & & & & & & & & & & \downarrow \text{Id} & & \\ Q_{s+1}^B & \longrightarrow & Q_s^B & \longrightarrow & Q_{s-1}^B & \longrightarrow & \cdots & \longrightarrow & Q_1^B & \longrightarrow & Q_0^B & \xrightarrow{f_0^B} & S_h & \longrightarrow & 0 \\ & & & & & & & & \searrow & & \nearrow & & & & \\ & & & & & & & & & \text{rad } Q_0^B & & & & & \end{array}$$

Since  $Q_0^B$  and  $S_h$  are  $B$ -module of finite type,  $f_0^B$  is an epimorphism and  $\text{Ker } f_0^B = \text{rad } Q_0^B$ , then  $f_0^B$  is an essential epimorphism in  $B$ . Because  $\text{rad}_A Q_0^B = \text{rad}_B Q_0^B$ , and the functorial immersion of the corresponding category modules, it follows that  $f_0^B$  is an essential epimorphism in  $A$ . In analogous way, if we consider the epimorphism  $\varphi_i^B : Q_i^B \longrightarrow \text{Ker } f_{i-1}^B$  in  $\text{mod } B$ , as  $\text{Ker } \varphi_i^B \subset \text{rad } Q_i^B$ , then  $\varphi_i^B$  is essential in  $B$  and therefore also on  $A$ .

Moreover, as  $Q_0^A$  is a projective  $A$ -module, then there is a morphism of  $A$ -modules  $h_0 : Q_0^A \longrightarrow Q_0^B$  such that  $f_0^B h_0 = f_0^A$ , i.e., the following diagram is commutative.

$$\begin{array}{ccc} & Q_0^A & \\ \swarrow h_0 & \downarrow f_0^A & \\ Q_0^B & \xrightarrow{f_0^B} & S_h \longrightarrow 0 \end{array}$$

Since  $f_0^B$  is essential in  $A$  and  $f_0^B h_0 = f_0^A$  is an epimorphism, it follows that  $h_0$  is an epimorphism. Then there is a morphism of  $A$ -module  $\widetilde{h}_0 : \text{Ker } f_0^A \rightarrow \text{Ker } f_0^B$  such that the following diagram with exact rows is commutative:

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & \text{Ker } f_0^A & \xrightarrow{\psi^A} & Q_0^A & \xrightarrow{f_0^A} & S_h & \longrightarrow & 0 \\
 & & \downarrow \widetilde{h}_0 & & \downarrow h_0 & & \downarrow Id & & \\
 0 & \longrightarrow & \text{Ker } f_0^B & \xrightarrow{\psi^B} & Q_0^B & \xrightarrow{f_0^B} & S_h & \longrightarrow & 0
 \end{array}$$

Furthermore, it is clear that  $\widetilde{h}_0$  is an epimorphism.

Consider the diagram

$$\begin{array}{ccccc}
 Q_1^A & \xrightarrow{\quad} & Q_0^A & & \\
 \downarrow \varphi^A & \nearrow f_1^A & \downarrow h_0 & & \\
 & \text{Ker } f_0^A & & & \\
 \downarrow \widetilde{h}_0 & & & & \\
 Q_1^B & \xrightarrow{f_1^B} & Q_0^B & & \\
 \downarrow \varphi^B & \nearrow \psi^B & \downarrow & & \\
 & \text{Ker } f_0^B & & & \\
 & \searrow & & & \\
 & 0 & & & 
 \end{array}$$

For the epimorphism  $\varphi^B : Q_1^B \rightarrow \text{Ker } f_0^B$ , there exists a morphism of  $A$ -modules  $h_1 : Q_1^A \rightarrow Q_1^B$  (since  $Q_1^A$  is projective module) such that  $\varphi^B h_1 = \widetilde{h}_0 \varphi^A$  and  $f_1^B h_1 = \psi^B \varphi^B h_1 = \psi^B \widetilde{h}_0 \varphi^A = h_0 \psi^A \varphi^A = h_0 f_1^A$ .

Repeating the arguments, we get the following commutative diagram with exact rows:

$$\begin{array}{ccccccccccc}
 0 & \longrightarrow & Q_s^A & \xrightarrow{\quad} & Q_{s-1}^A & \longrightarrow & \cdots & \longrightarrow & Q_1^A & \xrightarrow{f_1^A} & Q_0^A & \xrightarrow{\quad} & S_h & \longrightarrow & 0 \\
 & & \downarrow h_{s+1} & & \downarrow h_s & & & & \downarrow h_1 & & \downarrow h_0 & & \downarrow Id & & \\
 & & & & & & & & & & & & & & \\
 & & & & \text{Ker } f_{s-1}^A & & & & & & & & & & \\
 & & & & \downarrow \widetilde{h}_{s-1} & & & & & & & & & & \\
 P_{s+1}^B & \longrightarrow & Q_s^B & \xrightarrow{f_s^B} & Q_{s-1}^B & \longrightarrow & \cdots & \longrightarrow & Q_1^B & \xrightarrow{f_1^B} & Q_0^B & \xrightarrow{\quad} & S_h & \longrightarrow & 0 \\
 & & \downarrow \varphi_s^B & & \downarrow \psi_s^B & & & & & & & & & & \\
 & & & & \text{Ker } f_{s-1}^B & & & & & & & & & & 
 \end{array}$$

where  $h_i$  are epimorphisms. Therefore,  $Q_{s+1}^B = 0$  and  $pd_B S_h \leq pd_A S_h \leq n - 1$ . Then the algebra  $B$  has global dimension equal to  $n - 1$ , since  $pd_B \text{rad } Q_0 = n - 1$ . From Proposition 3.8, it follows that there is  $S_{j_1}$ , a composition factor of  $\text{rad } Q_0$ , such that  $pd_B S_{j_1} = n - 1$ . Consequently,  $pd_A S_{j_1} = n - 1$ .

For  $1 \leq i \leq n - 1$ , consider  $J_i = \{h \in Q_0 : pd_A S \geq n - i\}$  and  $B_i = A/\langle\{e_h : h \in J_i\}\rangle$ . A similar argument to the one used previously shows that  $\text{gl.dim. } B_i = n - (i + 1)$ . Therefore, there is a simple  $A$ -module  $S_{j_{i+1}}$  such that  $pd_A S_{j_{i+1}} = n - (i + 1)$ , and the proof is complete.  $\square$

Note that the dual statements of Propositions 3.2, 3.5, 3.8, Lemmas 3.6, 3.7 and Theorem 3.9 hold.

#### 4. CRITICAL ALGEBRAS.

In this Section we introduce a new family of algebras of global dimension three, the critical algebras. We characterize these algebras by quivers with relations. Finally, we show that every strongly simply connected schurian algebra, having global dimension at least three, must contain a critical algebra.

**Definition 4.1.** Let  $B$  be an algebra. We say that  $B$  is *critical* if it satisfies the following properties:

- i)  $B$  has a unique source  $i$  and a unique sink  $j$ ,
- ii)  $pd S_i = id S_j = 3$  and if  $S$  is a different simple  $B$ -module we have that  $pd S \leq 2$  and  $id S \leq 2$ ,
- iii) Let  $0 \rightarrow Q_3 \rightarrow Q_2 \rightarrow Q_1 \rightarrow Q_0 \rightarrow S_i \rightarrow 0$  be the minimal projective resolution of the simple  $S_i$  and let  $Q = \bigoplus_{k=0}^3 Q_k$ . Then all indecomposable projective  $B$ -modules are in  $\text{add } Q$ , and each indecomposable projective  $B$ -module is a direct summand of exactly one  $Q_k$ ,
- iv) Let  $0 \rightarrow S_j \rightarrow I_0 \rightarrow I_1 \rightarrow I_2 \rightarrow I_3 \rightarrow 0$  be the minimal co-injective resolution of simple  $S_j$  and let  $I = \bigoplus_{k=0}^3 I_k$ . Then all indecomposable injective  $B$ -modules are in  $\text{add } I$ , and each indecomposable injective  $B$ -module is a direct summand of exactly one  $I_k$ .
- v)  $B$  does not contain any proper full subcategory that verifies i), ii), iii) and iv).

Note that a critical algebra  $B$  has global dimension three. In addition,  $B$  is minimal in the following sense:  $B$  does not contain any proper full subcategory  $B$  whose global dimension is three.

**Theorem 4.2.** Let  $A = kQ/I$  be a strongly simply connected schurian algebra. Let  $i, j$  be vertices of  $Q$  and let

$$\cdots \longrightarrow Q_3 \longrightarrow Q_2 \longrightarrow Q_1 \longrightarrow P_i \longrightarrow S_i \longrightarrow 0$$

be the minimal projective resolution of  $S_i$  in  $A$ , and let  $P_j$  be a direct summand of  $Q_3$ . If  $C = \text{Conv}(i, j)$ , then there exists a projective  $C$ -module  $P$  such that  $\Gamma = \text{End}_C P$  is critical.

*Proof.* Let

$$0 \longrightarrow P_j^\alpha \longrightarrow \bigoplus_{k=1}^t P_{b_k}^{\alpha_k} \longrightarrow \bigoplus_{k=1}^u P_{a_k} \longrightarrow P_i \longrightarrow S_i \longrightarrow 0$$

be the minimal projective resolution of  $S_i$  in  $C$ . Consider

- $\mathcal{R} = \{b \in Q_0 : P_b \text{ is a direct summand of } Q_2 \text{ and there is a nonzero path } b \rightsquigarrow j\}$ ,
- $\mathcal{S} = \{a \in Q_0 : P_a \text{ is a direct summand of } Q_1, \text{ there are } b \in \mathcal{R} \text{ and } \rho \in I \text{ such that } a \rightsquigarrow b \text{ is a nonzero path and } \rho : a \rightsquigarrow j\} = \{a_1, \dots, a_s\}$ ,

Assume that, for  $1 \leq k \leq v$ ,  $\rho_k : a_k \rightsquigarrow j$ , there are commutativity relations going through  $b_k, b'_k$ , and that  $v+1 \leq k \leq s$ ,  $\rho_k$  are monomial relations that go through  $b_k$ .

Consider the projective  $C$ -module

$$P = P_i \oplus \bigoplus_{a \in \mathcal{S}} P_a \oplus \bigoplus_{b \in \mathcal{R}} P_b \oplus P_j$$

By abuse of language, we also call  $i, a_1, \dots, a_s, b_1, \dots, b_r, j$  the vertices of  $\Gamma$ , where  $r = \text{Card } \mathcal{R}$ .

Then in  $\Gamma = kQ_\Gamma/I_\Gamma$  we have:

- ★<sub>1</sub>.  $(Q_\Gamma)_0 = \{i, a_1, \dots, a_s, b_1, \dots, b_r, j\}$
- ★<sub>2</sub>. there is a only source  $i$  and only sink  $j$
- ★<sub>3</sub>. arrows starting at  $i$  are  $i \longrightarrow a_k$ ,  $1 \leq k \leq s$
- ★<sub>4</sub>. arrows ending at  $j$  are  $b_h \longrightarrow j$ ,  $1 \leq h \leq r$
- ★<sub>5</sub>. there are arrows  $a_k \longrightarrow b_h$ , for  $1 \leq k \leq s, 1 \leq h \leq r$ , where there are nonzero paths  $a_k \rightsquigarrow b_h$  in the algebra  $C$
- ★<sub>6</sub>.  $(Q_\Gamma)_1$  consists of all the arrows mentioned in ★<sub>3</sub>., ★<sub>4</sub>. and ★<sub>5</sub>..
- ★<sub>7</sub>. for each  $1 \leq h \leq r$ , there is a relation  $i \rightsquigarrow b_h$  (because this relation exists in  $C$ )
- ★<sub>8</sub>. for each  $1 \leq k \leq s$ , there is a relation  $a_k \rightsquigarrow j$  (because this relation exists in  $C$ )
- ★<sub>9</sub>.  $I_\Gamma$  is generated by the relations given in ★<sub>7</sub>. and ★<sub>9</sub>.
- ★<sub>10</sub>.  $pd_\Gamma S_j = 0$  and  $id_\Gamma S_i = 0$ , since  $j$  is the sink and  $i$  is the source in  $\Gamma$ .
- ★<sub>11</sub>.  $pd_\Gamma S_{b_h} = 1$  for  $1 \leq h \leq r$  and  $id_\Gamma S_{a_k} = 1$  for  $1 \leq k \leq s$ , since there are no relations starting at  $b_h$  and no relations ending at  $a_k$  in  $\Gamma$ .
- ★<sub>12</sub>.  $pd_\Gamma S_{a_k} = 2$  for  $1 \leq k \leq s$  and  $id_\Gamma S_{b_h} = 2$  for  $1 \leq h \leq r$ , since there are minimal relations starting at  $a_k$ , and there are minimal relations ending at  $b_h$  in  $\Gamma$ .

Then a projective resolution of  $S_i$  in  $\Gamma$  is of the form

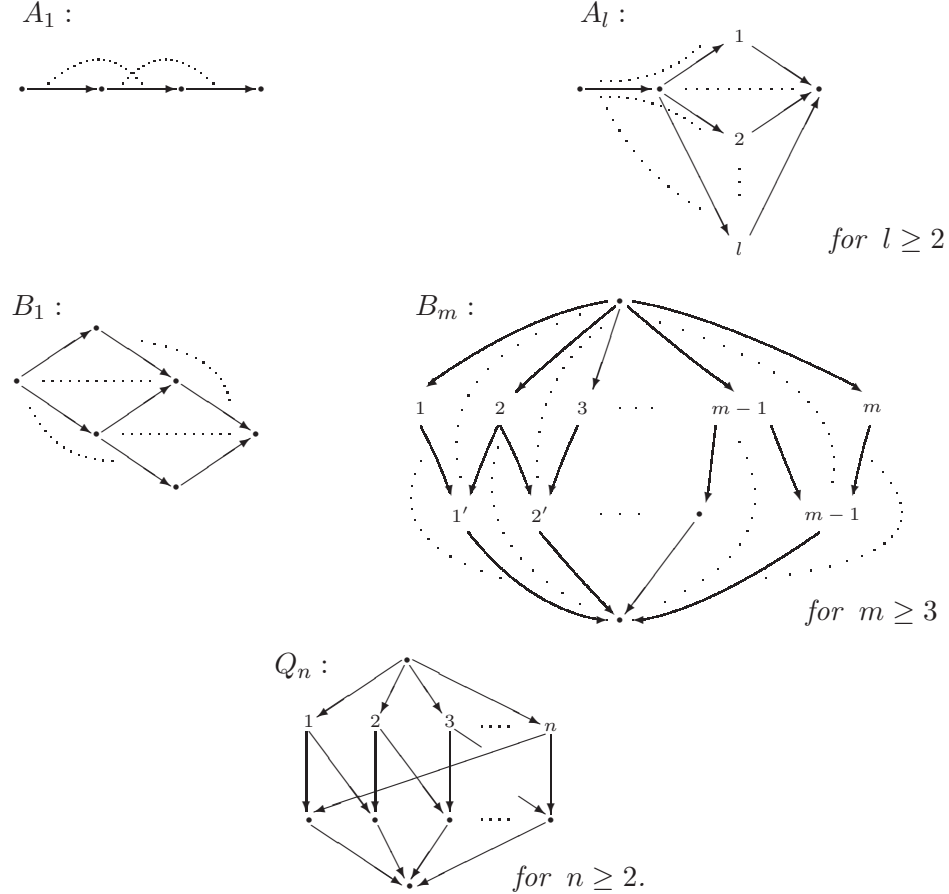
$$0 \longrightarrow Q_3^\Gamma \longrightarrow \bigoplus_{k=1}^s P_{b_k} \oplus \bigoplus_{k=1}^v P_{b'_k} \xrightarrow{f_2} \bigoplus_{k=1}^s P_{a_k} \xrightarrow{f_1} P_i \xrightarrow{f_0} S_i \longrightarrow 0$$

Since  $C$  is triangular, so is  $\Gamma$ . Consequently, the projective modules that occur as direct summands of the terms  $Q_0^\Gamma = P_i$ ,  $Q_1^\Gamma = \bigoplus_{k=1}^s P_{a_k}$ ,  $Q_2^\Gamma = \bigoplus_{k=1}^s P_{b_k}^{\alpha_k} \oplus \bigoplus_{k=1}^v P_{b'_k}^{\alpha'_k}$  can not appear as direct summands of  $Q_3^\Gamma$ . Then either  $Q_3^\Gamma = 0$  or  $Q_3^\Gamma = P_j^\alpha$ .

We show that  $Q_3^\Gamma \neq 0$ . In fact, since  $\mu_{Q_1^\Gamma}(S_j) = v$  and  $\mu_{Q_2^\Gamma}(S_j) = s + v$ , then  $\mu_{\text{Ker } f_2}(S_j) \geq s > 0$ . Since the relations  $\rho_k : a_k \rightsquigarrow j$  are minimal relations in  $\Gamma$ , for each  $1 \leq k \leq s$ , it follows that  $S_j \in \text{Top Ker } f_2$ . Therefore,  $pd_\Gamma S_i = 3$ , and hence  $\text{gl.dim. } \Gamma = 3$  and  $id_\Gamma S_j = 3$ . Then  $\Gamma$  is critical.  $\square$

We give below a description by quivers with relations of all critical algebras.

**Proposition 4.3.** *Let  $\Gamma$  be a critical algebra. Then either  $\Gamma$  or  $\Gamma^{op}$  is one of the following algebras.*





*Proof.* Let  $\Gamma = kQ/I$  be a critical algebra. From the proof of the Theorem 4.2 and the properties *iii*) and *iv*) of the critical algebras, it follows that there is the following partition in the set of vertices of  $\Gamma$ :

$$Q_0 = \{i\} \cup \{a_1, \dots, a_s\} \cup \{b_1, \dots, b_r\} \cup \{j\}$$

such that:

- there are  $i \longrightarrow a_k \in Q_0$ , for all  $1 \leq k \leq s$ ,
- there are arrows of the form  $a_k \longrightarrow b_h$ ,
- there are  $b_h \longrightarrow j \in Q_0$ , for all  $1 \leq h \leq r$ ,
- for each  $1 \leq h \leq r$ , there is a relation  $i \rightsquigarrow b_h$ .
- for each  $1 \leq k \leq s$ , there is a relation  $a_k \rightsquigarrow j$ ,

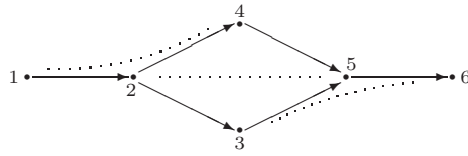
Then, varying  $s$  and  $r$ , and using the property *v*), we get all the quivers with relations for  $\Gamma$ . These quivers are the ones listed above, or their opposite quivers.  $\square$

We are now able to state our main theorem.

**Theorem 4.4.** *Let  $A = kQ/I$  be a strongly simply connected schurian algebra with  $\text{gl.dim. } A \geq 3$ . Then there exists a full subcategory  $B$  of  $A$  such that  $B$  is critical.*

*Proof.* Since  $\text{gl.dim. } A \geq 3$ , by Theorem 3.9, there exists a vertex  $i \in Q_0$  with  $\text{dp } S_i = 3$  and a vertex  $j \in Q_0$  such that  $P_j$  is a direct summand of the term  $Q_3$  of the minimal projective resolution of  $S_i$ . Consider  $\text{Conv}(i, j)$  and apply Lemma 3.4 and Theorem 4.2 to obtain the desired result.  $\square$

**Example 4.5.** Let  $A$  be the algebra given by the following quiver with relations

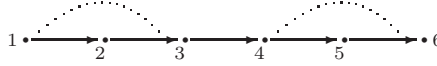


This algebra is strongly simply connected and schurian with global dimension three. Note that  $A$  contains a critical algebra. In fact, if we consider  $B = \text{End}_A(P_1 \oplus P_2 \oplus P_5 \oplus P_6)$ , we get that  $B$  is a critical algebra, which is isomorphic to  $A_1$ .

It follows from Theorem 4.4, that if a strongly simply connected schurian algebra  $A$  does not contain a critical full subcategory, then  $\text{gl.dim } A \leq 2$ . Therefore, we give a sufficient condition for deciding if the algebra has global dimension two.

The following example shows that the converse of Theorem 4.4 does not hold in general. An algebra of global dimension two may have a critical full subcategory.

**Example 4.6.** Consider the algebra  $A = kQ/I$ , given by the following quiver with relations



The algebra  $A$  is strongly simply connected and schurian, with global dimension two. However, the subcategory  $B = \text{End}_A(P_1 \oplus P_2 \oplus P_5 \oplus P_6)$  is critical, isomorphic to  $A_1$ .

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